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Modelling the Process of the Compression of Distance Knitted Fabrics in the Aspect of 'Elastica Curves'

Abstract
A new model of compressing distance knitted fabrics was developed based on Euler-Leaf theory. The model differs from those previously presented, which can be solved by an integro-differential equation of the first order. The solution of the equation, which at the same time determines the shape of the bent rod, takes the form of elliptic integrals, which is in contrast to previous models of the trigonometric function form. Calculation algorithms were elaborated to determine the functional dependencies between the compression force and deflection, as well as an algorithm describing curves that represent the shape of the rod compressed. An experimental stand was built to verify the measuring method.

Key words: *distance knitted fabrics, compression process, Euler's theory, elastic curves.*

■ Introduction

In recent years one can notice growing interest in the technical application of distance knitted fabrics. Apart from the shape of distance knitted fabrics, described by their geometry, they should meet application requirements, including the mechanical properties, especially their load strength and compression susceptibility. One can wonder why the modelling of the compression process of distance knitted fabrics is so important. The answer is simple: in order to learn the phenomena of compressing distance knitted fabrics. Along with the compression process of these fabrics comes a series of physical, in particular mechanical, phenomena, such as the deflection of connectors forming the structure of knitted fabrics and the mutual interaction of the friction between adjacent connectors [1, 2]. While conducting experiments, one can notice the phenomenon of various forms of the deflections of single monofilament threads comprising the middle layer of the fabrics. The cognition of this phenomenon allows us to attempt to describe it mathematically, not verbally, bringing us closer to an understanding of this phenomenon.

The present article is a continuation of our previous research work. A number of mathematical models of the compression

process of distance knitted fabrics based on Euler's theory of the buckling process (with so called elastic slender rods) have already been described. The solution of Euler's differential equation is a combination of trigonometric functions and, at the same time, an image of the longitudinal axis of the monofilament compressed [3, 4]. This article presents a model of the compression of distance knitted fabrics based on Euler-Leaf's theory [5, 6], where the solution of Euler's equation is introduced by elliptic integrals. The object compressed, which is a distance knitted fabric, is considered as two rigid planes connected by a network of rods forming an elastic space truss. The shape of the truss depends on the knitted structure of the stitches, as well as on the mechanical properties of the threads forming the structure.

Research into the compression process of 3D distance knitted fabric is carried out in two ways: the shape analysis of a bent monofilament and mechanical analysis of a single monofilament forming the structure of the internal layer of a 3D distance knitted fabric. The final stage of our research work is the verification of the results obtained by theoretical methods with those obtained in experiments.

▌ Mathematical model to analyse the compression process of the monofilament connectors of the internal layer of a knitted fabric

As already mentioned in the introduction, the model of compressing distance knitted fabrics is based on the mechanical analysis of the process of bending a single element of the knitted fabric (a

connector creating the internal layer of the distance knitted fabric).

Leaf studied a similar problem, in which he attempted to find a mathematical model of a plain knitted loop. The principal on which Leaf's assumptions are based results from the shape analysis of a homogenous elastic bent rod. A perfectly straight rod buckled by equal and opposite forces applied at its ends takes a shape called 'elastica' [5]. The model of compressing distance knitted fabrics is based on the following assumptions:
- The bent object is considered as a slender rod made of a homogenous and elastic material.
- The rod is fastened at both ends to the external surfaces of the distance knitted fabric by an articulated joint.
- A straight line, led along the initial axis of the bent rod, goes through the ends of the rod. A force *P* also acts along this straight line.
- The process of compression is considered in the static mode.
- The rod is assumed to be a weightless body.
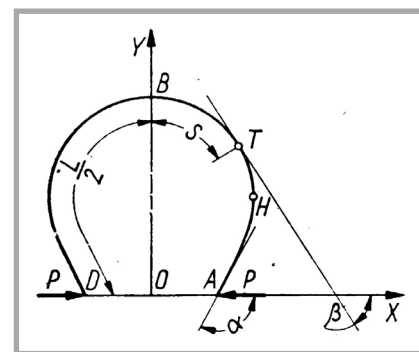- The bending of the rod occurs only in a single plane.

Figure 1. *Shape of the bent elastic rod analysed by Leaf.*

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The model of compressing distance knitted fabrics is based on Equation (1) – Euler's theory. The equation of the bent rod may be written as:

$$EJ \frac{1}{\rho} = -Py \qquad (1)$$

where:
EJ - bending rigidity,
$1/\rho$ - curvature of the rod at point T (**Figure 1**),
P - forces acting on the ends of the rod,
Y - distance between the force P (applied at end points of the rod) and point T,
E - Young modulus,
J - moment of inertia of the bent rod's cross-section.

With the geometrical relations presented in **Figure 1**, Equation (1) can be expressed in a integro-differential form:

$$EJ \frac{d\beta}{ds} + P \int_{L/2}^{s} \sin\beta\, ds = 0 \qquad (2)$$

where $\quad \dfrac{d\beta}{ds} = \dfrac{1}{\rho}, \quad \displaystyle\int_{L/2}^{s} \sin\beta\, ds = y$

Differentiating Equation (1) with respect to s, we finally obtain a differential equation of first order for the deflection line:

$$\frac{d^2\beta}{ds^2} + \frac{P}{EJ} \sin\beta = 0 \qquad (3)$$

Introducing the auxiliary function $\Omega \equiv \dfrac{d\beta}{ds}$, we multiply both sides of Equation (3) by Ω:

$$\Omega \frac{d\Omega}{ds} + \frac{P}{EJ} \sin\beta \frac{d\beta}{ds} = 0 \qquad (4)$$

Multiplying by ds, we obtain:

$$\frac{1}{2} d(\Omega^2) + \frac{P}{EJ} \sin\beta\, d\beta = 0 \qquad (5)$$

After integrating relation (5) and extracting the root, we get:

$$\pm\Omega \equiv \pm\frac{d\beta}{ds} = \sqrt{\frac{2P}{EJ}\cos\beta + C} \qquad (6)$$

We determine integration constant C from the following set of boundary conditions: for s = $(l_p = L)/2$, angle $\beta = \alpha$, and y = 0,
therefore $\dfrac{1}{\rho} = 0$, that is $\dfrac{d\beta}{ds} = 0$, we get:

$$C = -\frac{2P}{EJ}\cos\alpha \qquad (7)$$

Putting C in the equation for Ω, we find:

And using Equation (8), we get a separable variable equation:

$$\frac{d\beta}{ds} = \sqrt{\frac{2P}{EJ}(\cos\beta - \cos\alpha)} = \\ = \sqrt{\frac{4P}{EJ}\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\beta}{2}\right)} \qquad (8)$$

And using Equation (8), we get a separable variable equation:

$$ds = \frac{d\beta}{\sqrt{\dfrac{4P}{EJ}\left(\sin^2\dfrac{\alpha}{2} - \sin^2\dfrac{\beta}{2}\right)}} \qquad (9)$$

Double integration with respect to the half length of the rod leads to:

$$s\sqrt{\frac{P}{EJ}} = \int_{0}^{\frac{\beta}{2}} \frac{d\left(\dfrac{\beta}{2}\right)}{\sqrt{\dfrac{4P}{EJ}\left(\sin^2\dfrac{\alpha}{2} - \sin^2\dfrac{\beta}{2}\right)}} \qquad (10)$$

Where, half of the length of the rod is considered as a function, while the other half is a mirror image of the first half with respect to the Y-axis.

To bring the integral to a Legendare canonical form, first we substitute:

$$\sin\frac{\beta}{2} \equiv \sin\frac{\alpha}{2}\sin\psi,$$

where $\quad \psi = arc\sin\dfrac{\sin\dfrac{\beta}{2}}{\sin\dfrac{\alpha}{2}}$

Which gives:

$$\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\beta}{2}} = \sin\frac{\alpha}{2}\sin\psi \qquad (11)$$

For the element of integration (10), we have $s = l_p/2$, $\psi = \pi/2$ in point A, where $\beta = \alpha$, whereas in point B, when $\beta = \pi$, $\psi = 0$, we obtain:

$$l_P = 2\sqrt{\frac{EJ}{P}} \int_{0}^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1 - m\sin^2\psi}} = 2uF(\frac{\pi}{2}, m), \quad (12)$$

therefore $\quad F(\dfrac{\pi}{2}, m) = \dfrac{l_P}{2u},$

where $\quad u^2 = \dfrac{EJ}{P}, \quad m = \sin^2\dfrac{\alpha}{2}$

Function $F(\pi/2, m)$ is a complete elliptic integral of the first kind,
l_P - length of the deflected monofilament,
P - compression force.

The complete elliptic integral of the first kind is defined as:

$$F(\frac{\pi}{2}, m) = \int_{0}^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1 - m\sin^2\psi}} \qquad (13)$$

Where 'm' is a modulus, taking values from <0, 1>, therefore function $F(\pi/2, m)$ takes values of $< 1.57079, \infty)$. In order to determine the theoretical characteristic of monofilament compression, one has to determine the Young modulus 'E', evaluate the modulus of inertia "J" of the cross-section of the monofilament perpendicular to the longitudinal axis of the rod, and take any value of force P from the range $< 1.57079, \infty)$, for which one has to determine the deflection Δg. Using the values of 'E', 'J', 'P' and the length of the monofilament l_P, we can calculate the right side of Equation (12). With modulus 'm' already defined, we can now determine the deflection of the rod Δg and the deflection curve of the rod using the following formulas [2, 7, 8]:

$$\begin{aligned} x &= u[2E(\psi, m) - F(\psi, m)] \\ y &= 2u\sqrt{m}\cos\psi \end{aligned} \qquad (14)$$

Where x and y are the coordinates of any point of the bent rod's axis, $F(\psi, m)$ is an incomplete elliptic integral of the first kind

$$F(\psi, m) = \int_{0}^{\psi} \frac{d\psi}{\sqrt{1 - m\sin^2\psi}}, \qquad (15)$$

while $F(\psi, m)$ is an incomplete elliptic integral of the second kind

$$E(\psi, m) = \int_{0}^{\psi} \sqrt{1 - m\sin^2\psi}\, d\psi. \qquad (16)$$

If we assume that for non-deformed fabric ($P = 0$) the rod tested is a line segment of length $l_P = g$, where 'g' is the thickness of the non-deformed fabric, then $\Delta g = g - 2x$.

This results from the fact that the formulas present a parametric equation of the curve, which is the right side (the positive side of the X-axis) of the curve presented in **Figure 1**; the left side of the curve is a mirror image of the right side of the curve with respect to the Y-axis. Therefore, when calculating the deflection Δg, we double the value of the coordinates of the monofilament's ends 'x' (point A). The value of parameter ψ for point A – the end of the rod for which the deflection Δg is calculated – equals $\psi = \pi/2$, as in point A $\alpha = \beta$. In point A, therefore, the upper limit of integrating (15) and (16) equals $\psi = \pi/2$, and the integrals may be written as $F(\pi/2, m)$ and $E(\pi/2, m)$.

With modulus "m" known and functions $F(\psi, m)$ and $E(\psi, m)$ taking the shape of definite integrals of known limits of
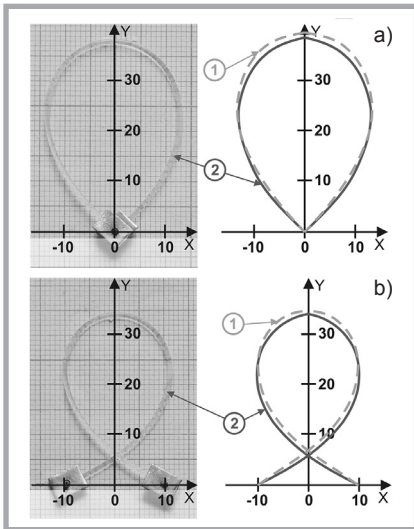
Figure 2. *Photographs and shapes of the bent rod obtained by theoretical and experimental methods; for a) P = 0.142 N, b) P = 0.2 N, 1 – experimental curve, 2 – theoretical curve.*

integration, we can now determine coordinate 'x' using the first formula from (14). Putting the value 'x' in the following equation: $\Delta g = g - 2x$, we obtain the deflection Δg that corresponds with force P. With modulus 'm' known, we can also determine the deflection curve of the rod using formulas (14), which define the parametric curve, where ψ is the parameter, changing within the range $\psi \in (0, \pi/2)$. In order to determine the deflection curve of the rod, one should identify a sequence of points (x, y) for parameter ψ from equation (14). The deflection curve of a monofilament compressed by force P is presented in **Figure 2** as a theoretical curve for 10 analytical points. To verify the real shapes of the bent rod obtained by theoretical methods, experimental tests were carried out in order to determine its shape. As a result of the experimental research, we obtained the

theoretical and experimental characteristics of the bent rod, presented as a set of points. **Figure 3** presents a photograph of a stand for the shape registration of bent polyamide monofilament, while **Figure 2** presents a photograph of the real shape of the compressed rod as well as a diagrammatic presentation of the photograph (curve 1), and theoretical curve (curve 2).

Mechanical characteristics of a single monofilament

In order to use Leaf theory for modelling the compression process of the rod as a single monofilament of 3D distance knitted fabric, one has to assume that the compressed rod is placed between two rigid planes and fastened at both ends by articulated joints. Therefore, the theory mentioned above might be used till α, the angle between the tangent to the deflection curve at point A (**Figure 1**) and the X-axis, reaches the value of $\alpha = \pi/2$ (the deflection Δg_2). Further compression of the fabric does not cause a change in angle $\alpha = \pi/2$ (deflections Δg_3 and Δg_4). The deflection of the rod causes changes in its length l_P. The mechanical model of the bent rod based on Euler-Leaf theory is presented in **Figure 4**.

The parallel displacement of the application point of force P is an effect of the phenomenon (the part of the rod contiguous to the external surfaces of the fabric does not participate in the compression process. The ends of the monofilament undergo a parallel displacement). In order to create a further part of the compressed rod characteristic (which is the internal layer element of 3D distance knitted fabric) for the values of forces, where angle α would exceed the value
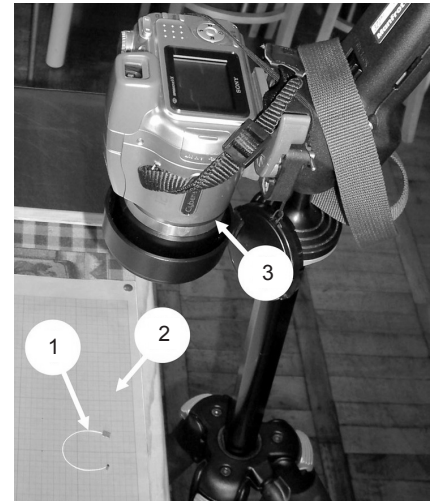

Figure 3. *Photograph of a stand for the shape registration of bent monofilament: 1 – monofilament, 2 – graph paper, 3 – digital camera.*

$\alpha = \pi/2$, we put this value into the following formula:

$$m = (\sin\frac{\alpha}{2})^2 = (\sin\frac{\pi}{4})^2 = \frac{1}{2} \quad (17)$$

determining modulus 'm' and consequently integral (13), whose value from now on is known and constant. Equation (12) may be written as $P = f(l_P)$ as the other values (E, J and $F(\pi/2, m)$) are known and constant. The equation may be written as:

$$P = \frac{1}{(l_P)^2}4EJ[F(\frac{\pi}{2},m)]^2 \quad (18)$$

Value l_P is treated as an independent variable whose value is reduced below the initial one. We divide the range $(0, l_P)$ into 9 parts and obtain the following sequence of values:

$l_P = l_P/9, 2l_P/9, 3l_P/9, 4l_P/9, 5l_P/9, 6l_P/9, 7l_P/9, 8l_P/9, l_P$, which determines the sequence of values of force P_i, where $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$. With modulus "m" defined, we can now determine the deflection of the rod, Δg, and the deflection curve of the rod for modulus 'm' for each value of force P_i. The method of determining the deflection of the rod, Δg, and the deflection curve of the rod has already been presented, taking the value of force P_i instead of P in formulas (14). The theory described helps to determine the theoretical characteristic of compressing a rod fastened at both ends by articulated joints. The parameters of the compression process are the following: $EJ = 67$ N·mm² - rigidity of the polyamide rod compressed, $l_P = 100$ mm
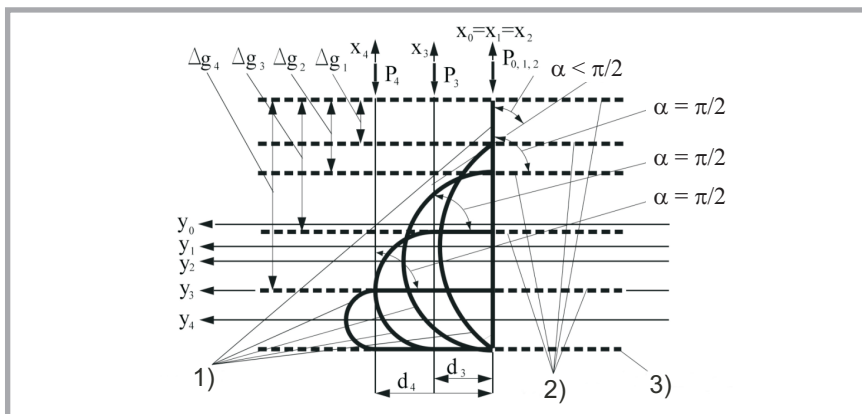

Figure 4. *Mechanical model of the bent rod based on Euler-Leaf theory. 1 – deflected monofilament, 2 – top layer (movable), 3 – bottom layer (immovable).*
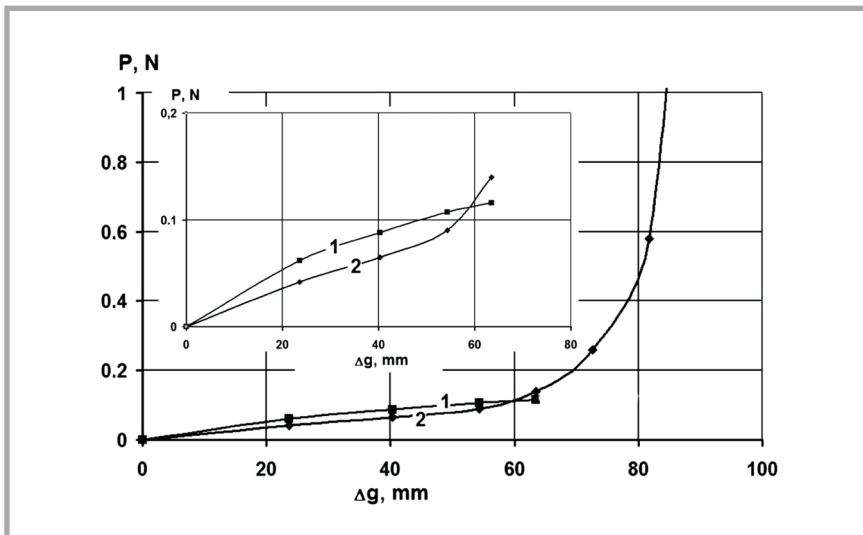
Figure 5. Mechanical characteristics of the bent rod: 1 – empirical, 2 – theoretical.

- length of the rod, $\emptyset$ = 1 mm - diameter of the rod.

An algorithm determining the characteristic of compression and one determining the theoretical shape, in order to compare it with the real shape, were elaborated using *"Mathematica"* software. **Figure 5**. presents characteristics of the rod compressed. We also conducted a research study in order to verify the method of determining the mechanical characteristics of the compressed rod in 3D distance knitted fabric (**Figure 6**).

The measuring stand for determining the mechanical characteristics of bent monofilament is adapted for testing the polyamide threads in articulated joints. The force bending the monofilament thread is measured by the strain gauge. The voltage related to the bending force is measured by the sensitive voltmeter. In order to visualise the measuring process, the measuring stand is equipped with a
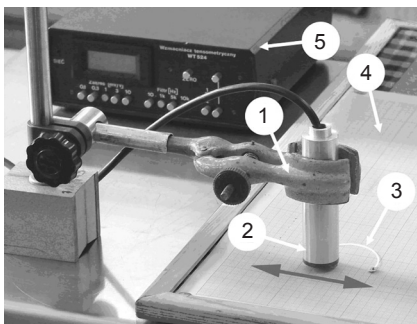

Figure 6. Photograph of a experimental stand for the determination of the mechanical characteristics of bent monofilament: 1 – stand/support, 2 – strain gauge, 3 - monofilament, 4 – graph paper, 5 – amplifier and voltmeter.

digital camera. Graph paper is used as the background of the sample in the deflection measurement of monofilament thread.

The comparative analysis of a single rod compression shows the differences between the calculated and experimental values, which do not exceed 18%. The differences between the results obtained from the mathematical model of the compression process and those obtained from the experimental model occur till the deflection values do not exceed Δg = 62 mm. Within the range of deflection values established, there is a linear dependence between the deflections and deflecting forces; however, after the value mentioned is reached comes a strong non-linear increase in the deflecting forces. The results obtained, referring to both the model shapes of the rod compressed as well as the mechanical characteristics of the compression process, allow for a more optimistic attitude towards the model of the compression process presented above.

■ Conclusions

1. A model of the compression of distance knitted fabrics was developed based on the theory of compressing an elastic slender rod according to Euler-Leaf theory, the assumptions of which are presented. The mathematical model is described by an integro-differential equation of the first order, with a function of bending the rod's shape in the form of an elliptic integral as its solution. The model helps to elaborate the characteristics of the

monofilament compression of a fabric's internal layer. The model elaborated is just an introduction to further simulative research into compressing 3D distance knitted fabrics, taking into consideration the mechanical parameters of monofilaments, the structural parameters of knitted fabrics, and weave parameters.

2. In the aspect of determining the deflection curves of the rod, the model of the compression process was empirically verified by the real shapes measured using digital image analysis techniques. The theoretical and experimental curves of the bent rod (for different aspects of deflection) have a high shape compatability. The discrepancy of measurements are simply the errors of the measurements and methods used.

3. Simulation of the compression process of distance knitted fabric was carried out and the results compared with real compression characteristics. The experimental research was conducted on a measuring stand, using a strain gauge force meter. The differences between the values of theoretical and experimental forces do not reach the limit of 18%, which may be explained by the contracted program of experimental research as well as by the errors of the methods.

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