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An Analysis of Young's Modulus Distribution in the Paper Plane

Abstract

Within our research work we determined the Young's modulus distribution in the paper plane on the basis of results of tensile tests in selected directions. Using mathematical relationships valid for orthotropic materials, the moduli were calculated theoretically. The tensile stiffness orientation angle (TSA), and the TSO and TSI indices were also determined in order to define the deviations of the main axes of orthotrophy in the paper plane as determined by the machine and the cross directions. The calculation results were compared with measured values. They justified the conclusion that paper may be treated as an orthotropic material.

Key words: *paper, Young's modulus, orthotrophy.*

■ Introduction

Paper is a fibrous material whose structure depends both on the method of manufacture and the materials used.

Generally paper may be considered as a composite containing fibres and a matrix consisting of filling materials and other paper-auxiliary agents.

The weight share of fibres in paper ranges from 70% to 95% on average. The most appropriate fibres for paper production are fibres of a length from 1 mm to 5 mm. It has been assumed that the fibre slenderness (the ratio of the length to the thickness) should be higher than 40:1. However, the strength properties of the manufactured fibrous structure are more influenced by the ratio of the thickness to the lumen diameter than by the slenderness. The lower the ratio is, the better the strength properties which can be attributed to paper.

Paper has a porous structure, in which fibres form a three-dimensional net. This is why, despite its tensile strength being comparable to that of steel, the value of Young's moduli for paper is two orders of magnitude lower than for steel. In many cases, both in practical applications and laboratory tests, paper is considered as an elastic material. Young's modulus is one of standardised mechanical properties as determined for paper. There are also other methods for analysing paper properties such as Poisson's ratio [1, 2, 4] or the modulus of rigidity [1, 3, 4].

The well-known ultrasonic method, used for determining Young's modulus, is also used as a measuring method for testing paper properties. Recently it has grown in popularity for determining the tensile stiffness index (TSI) and the tensile stiffness orientation angle (TSO) [5]. The method allows those paper properties which are significant from a practical point of view to be quickly determined.

The opinions promoted by some researchers [6], namely that in the stress range of up to 20% of breaking stress permanent deformations do not appear, does not seem to be completely true, although in the majority of practical applications with low stress and short duration time, the error is rather minor when using the theory of linear stress to measure the paper behaviour.

Numerous research works have analysed unidirectional linear stress in paper plane. Such an approach is too simplistic, and does not allow some aspects connected with paper making, conversion and utilisation to be analysed.

Undoubtedly over 90% of the paper loads which arise in practice are connected with the forces acting in plane, and the stress usually has a complex nature. For this reason, from a practical point of view, analysis of the stress perpendicular to the paper plane is not necessary, and the paper can be considered as a thin film exposed to breaking strength.

Paper is a fibrous material, and fibre orientation in the web depends on the method of manufacture. Papers made in industrial conditions (on fourdrinier machines) show an oriented distribution of their mechanical properties, due to the fibre orientation in the paper. The main axes of this distribution, for which extreme values of breaking stresses, deformations at the moment of break and Young moduli are achieved, overlap with machine (MD) and cross direction (CD), or they vary with a minor angle.

$$\frac{1}{E_{x\alpha}} = \frac{\cos^4\alpha}{E_x} + \left(\frac{1}{G_{xy}} - \frac{2\nu_{xy}}{E_x}\right)\sin^2\alpha\,\cos^2\alpha + \frac{\sin^4\alpha}{E_y} \quad (5)$$

$$\left[\frac{1}{E_{x\alpha}} - \frac{\cos^4\alpha}{E_x} - \frac{\sin^4\alpha}{E_y}\right]\cdot\frac{1}{\sin^2\alpha\,\cos^2\alpha} = \frac{1}{G_{xy}} - \frac{2\nu_{xy}}{E_x} \quad (6)$$

$$\left[\frac{1}{E_{x\alpha_1}} - \frac{\cos^4\alpha_1}{E_x} - \frac{\sin^4\alpha_1}{E_y}\right]\cdot\frac{1}{\sin^2\alpha_1\cos^2\alpha_1} = \frac{1}{G_{xy}} - \frac{2\nu_{xy}}{E_x} \quad (7)$$

$$s = \frac{\left|\dfrac{dE_{x\alpha}}{dE_x}\right|\cdot\Delta E_x + \left|\dfrac{dE_{x\alpha}}{dE_y}\right|\cdot\Delta E_y + \left|\dfrac{dE_{x\alpha}}{dE_{x\alpha_1}}\right|\cdot\Delta E_{x\alpha_1}}{E_{x\alpha}}\cdot 100\% \quad (10)$$

$$E_{x\alpha} = -\frac{E_x\,E_{x\alpha_1}\,E_y \sin^2\alpha_1\,\cos^2\alpha_1}{E_{x\alpha_1} E_y \cos^4\alpha_1\ \sin^2\alpha\,\cos^2\alpha - E_{x\alpha_1}\sin^2\alpha_1\cos^2\alpha_1\,(E_y\cos^4\alpha + E_x\sin^4\alpha) + E_x\,\sin^2\alpha\,\cos^2\alpha\,(E_{x\alpha_1}\sin^4\alpha_1 - E_y)} \quad (8)$$

Equations: 5, 6, 7, 8, and 10.

The distribution of mechanical properties in the paper plane corresponds with properties characteristic of orthotropic materials. This thesis can be found in numerous academic publications [7 - 9].

Method of determining Young's modulus

Considering paper as a linearly elastic and orthotropic material, we can determine a relationship between the strains and stresses in the system of coordinates overlapping the directions of the main axes of orthotrophy (x, y) by the following formulas: [10]:

$$\varepsilon_x = \frac{1}{E_x}\sigma_x - \frac{\nu_{yx}}{E_y}\sigma_y \quad (1)$$

$$\varepsilon_y = -\frac{\nu_{xy}}{E_x}\sigma_x + \frac{1}{E_y}\sigma_y \quad (2)$$

$$\gamma_{xy} = \frac{1}{G_{xy}}\tau_{xy} \quad (3)$$

where:
ε_x, ε_y – strains,
σ_x, σ_y – normal stress,
E_x, E_y – Young moduli,
ν_{xy}, ν_{yx} – Poisson's ratios,
γ_{xy} – shear strain,
G_{xy} – shear modulus,
τ_{xy} – shear stress.

According to the conclusion of the symmetry of the stiffness matrix (or to the Betty theorem) the Young's moduli and Poisson's ratio have to satisfy the following relation:
$$E_x\nu_{xy} = E_y\nu_{xy} \quad (4)$$

The Young modulus $E_{x\alpha}$ for the axis at angle α to axis x can be determined

Table 1. Papers used in tests.

Paper grade	Basic weight, g/m²	Thickness, mm
Liner	220	0.347
Fluting	123	0.241
Coated paper	98	0.103

from the formula shown in Equation 5. After transformation, Equation 5 can be presented as Equation 6. Substituting α into any angle α_1 gives Equation 7. The right sides of formulas 6 and 7 are equal, and $E_{x\alpha}$ can be calculated based on Equation 8, which allows Young's modulus to be determined for any direction in the paper plane, provided the Young's moduli for the main directions of orthotrophy E_x, E_y are known, including that for any indirect direction $E_{x\alpha 1}$.

In order to examine the compatibility of the real distribution of Young's moduli in the paper plane with the modulus characteristic of orthotropic materials, its values for the papers listed in Table 1 were determined.

The moduli of elasticity were determined for seven directions in the paper plane. Measurements were made for longitudinal '0°', cross '90°' and five directions in-between, deviated from the longitudinal direction by 15°, 30°, 45°, 60° and 75°. The values of Young's modulus were defined, based on the stress-strain diagram produced by the paper in selected directions.

In addition, an ultrasonic examination was conducted [5] to determine the indices of tensile stiffness (TSI) and the tensile stiffness orientation angle (TSO).

The TSI can be compared to Young's modulus, because the velocity of an ultrasonic pulse propagating in the plane of the paper sheet corresponds with the sheet elastic properties:

$$v_x = \sqrt{\frac{E_x}{d}\cdot c} \approx \sqrt{\frac{E_x}{d}} \quad (9)$$

where:
v_x – propagation velocity,
δ – density,
E_x – Young's modulus,

c – constant depending on Poisson's ratio.

The TSO equals the value of the angle between MD $\equiv x$ and the direction of the TSI's maximum value (see Figure 1).

Based on the values of the moduli obtained for the longitudinal and cross directions, as well as those selected from other five indirect directions, the theoretical values of the Young moduli for the other four directions were calculated by formula (8).

Research results and analyses

Paper is a fibrous material, containing a large number of pores filled with air. The nature of this structure means that the change of paper thickness (for example, during the calendaring process) may not influence the variation of strength properties, as this change appears as a result of the decreasing pore volume, and when included in calculations of breaking stress or Young's modulus, significantly changes their value.
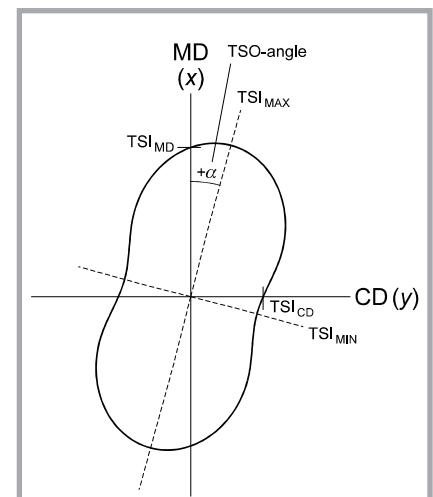
Figure 1. TSO diagram.

Table 2. *Young's moduli for liners; in cases marked with a framing in the table, the divergences between the values obtained during testing and calculations did not fall within the limits of the estimated errors.*

Method of obtaining results	Deviation of testing direction in relation to MD									
	15°		30°		45°		60°		75°	
	E, kN/m	s, %	E, kN/m	s, %	E, kN/m	s, %	E, kN/m	s, %	E, kN/m	s, %
Measurement	1855	1.8	1463	2.5	1172	2.5	992	2.1	809	2.7
Calculation*) 15°	—	—	1735	13.1	1409	17.9	1066	14.3	858	9.4
Calculation*) 30°	1740	3.6	—	—	1172	5.7	956	7.1	832	7.5
Calculation*) 45°	1739	4.7	1461	5.1	—	—	956	5.1	832	6.9
Calculation*) 60°	1777	7.4	1542	12.3	1244	9.7	—	—	841	6.1
Calculation*) 75°	1639	21.9	1266	45.0	1006	45.0	868	25.8	—	—
Measurement	$E_x \equiv E_{MD}$ = 1858 kN/m					$E_y \equiv E_{CD}$ = 793 kN/m				

* Calculation made on the basis of the value of Young's modulus for the direction sloped to MD at a given angle.

Table 3. *Young's moduli for fluting.*

Method of obtaining results	Deviation of testing direction in relation to MD									
	15°		30°		45°		60°		75°	
	E, kN/m	s, %	E, kN/m	s, %	E, kN/m	s, %	E, kN/m	s, %	E, kN/m	s, %
Measurement	799	2.3	702	1.3	587	1.7	512	2.7	449	2.7
Calculation*) 15°	—	—	744	14.3	645	18.6	542	12.9	472	5.1
Calculation*) 30°	782	3.2	—	—	603	3.3	519	3.3	465	2.4
Calculation*) 45°	775	3.9	684	4.5	—	—	509	2.2	463	1.9
Calculation*) 60°	776	5.4	688	6.8	590	5.9	—	—	464	2.2
Calculation*) 75°	735	10.2	600	18.3	505	19.2	461	12.1	—	—
Measurement	$E_x \equiv E_{MD}$ = 813 kN/m					$E_y \equiv E_{CD}$ = 449 kN/m				

* Calculation made on the basis of the value of Young's modulus for the direction sloped to MD at a given angle.

Table 4. *Young's moduli for coated paper.*

Method of obtaining results	Deviation of testing direction in relation to machine direction									
	15°		30°		45°		60°		75°	
	E, kN/m	s, %	E, kN/m	s, %	E, kN/m	s, %	E, kN/m	s, %	E, kN/m	s, %
Measurement	682	3.1	627	2.7	528	8.3	482	2.1	413	3.2
Calculation*) 15°	—	—	582	12.4	487	16.6	419	15.0	382	11.2
Calculation*) 30°	710	2.8	—	—	529	6.2	442	8.6	388	9.8
Calculation*) 45°	700	5.7	625	10.2	—	—	442	9.9	387	10.1
Calculation*) 60°	733	6.7	709	13.8	608	12.0	—	—	397	8.1
Calculation*) 75°	675	23.1	568	54.4	474	57.6	413	33.2	—	—
Measurement	$E_x \equiv E_{MD}$ = 728 kN/m					$E_y \equiv E_{CD}$ = 370 kN/m				

* Calculation made on the basis of the value of Young modulus for the direction sloped to MD at a given angle.

Table 5. *Measurement results for TSO and TSI.*

Index	Liner		Fluting		Coated Paper	
	Min	Max	Min	Max	Min	Max
TSO_{MD}, deg	0.07	3.35	−1.09	−0.93	1.71	2.1
TSI_{MD}, kN·m/g	10.74	10.87	9.09	9.36	9.86	10.31
TSI_{CD}, kN·m/g	4.83	4.86	4.62	4.80	5.86	5.98
TSI_{MIN}, kN·m/g	4.83	4.88	4.62	4.8	5.86	5.98
TSI_{MAX}, kN·m/g	10.77	10.87	9.10	9.36	9.86	10.32

For practical reasons when describing paper properties, tension is very often used as a measure of a specific load in plane (that is, force referring to the length unit), instead of stress (force referring to the surface area). The Young's moduli analysed below were expressed in the same way. To achieve Young's modulus in stress units, its value (referring to the length unit) must be divided by the paper's thickness.

The tensile trials were conducted in accordance with standard PN-EN ISO 1924-2. The only exception to the standard was made for the stretching speed, which is described as 20 mm/min, whereas we tested at 10 mm/min.

Tables 2 – 4 contain the results of the measurements and calculations of Young's moduli E for the individual papers, along with errors s. In the measurement results, the error s is a relative mean error of the arithmetical mean of measurements, in percent. When defining the error of calculated values, the error s was determined with the Equation 10,
where:

ΔE_x – the mean error of arithmetical mean for measurements of Young's modulus in the x-direction,

ΔE_y – the mean error of arithmetical mean for measurements of Young's modulus in the y-direction,

$\Delta E_{x\alpha}$ – the mean error of arithmetical mean for measurements of Young's modulus deviated by angle α from the x-direction.

When calculating Young's moduli and determining the calculation error, it was assumed that the main orthotrophy directions in-plane of the papers examined overlapped with the MD and CD. In reality, those directions do not overlap. This could be the reason why in two cases for the liners, the divergences between values obtained during testing and calculations do not fall within the estimated errors. Those cases are marked with a framing in the table. The divergences between the results obtained through measurements and calculations are slightly greater than the errors, and the difference between them does not exceed 1% of measured value.

Figure 2 (see page 94) shows the measurement and calculation results for fluting.

The greatest divergences between measurement and calculation results appeared when the Young's modulus was determined on the basis of measurement results for directions approximate to the main direction in the paper, i.e. for those directions which deviated from MD by 15° and 75°. This is connected with the large errors of calculations, such as those results presented in the last line of Table 2. These results were obtained on the basis of the Young modulus value, for inclination to the direction of MD under an angle of 75°. The most similar
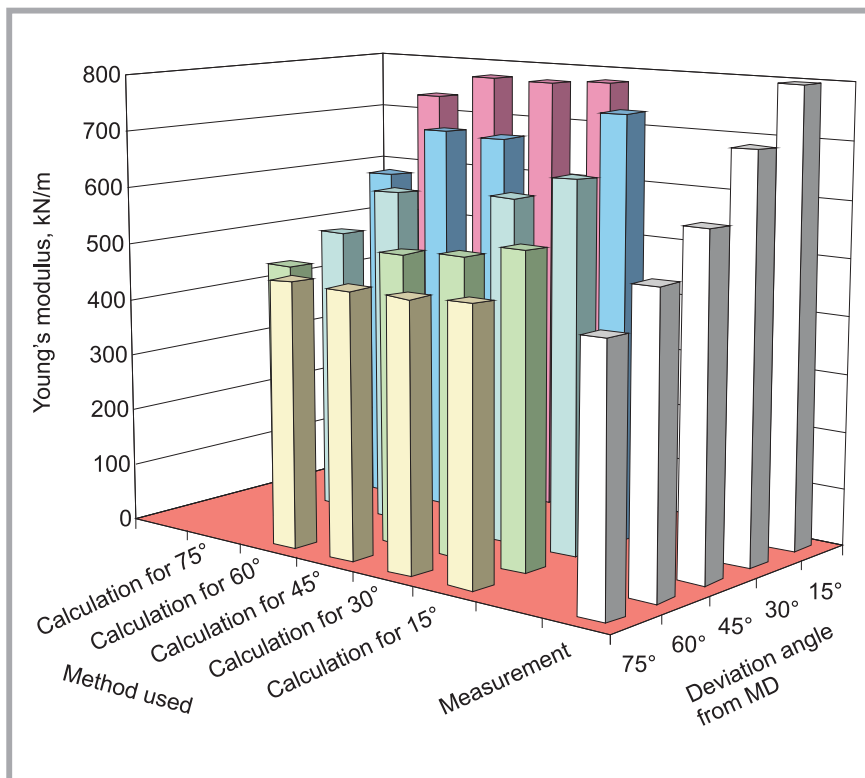
Figure 2. *Young's moduli for fluting.*

Table 6. *Comparison of Young's moduli*

Paper grade	Direction	E_{TSI}, kN/m	$E_{Tension}$, kN/m	$(E_{TSI} - E_{Tension})/E_{TSI}$, %
Liner	MD	2560	1858	27
	CD	1060	793	25
Fluting	MD	1130	813	28
	CD	585	449	23
Coated paper	MD	990	728	26
	CD	578	370	36

calculation results to the real values were achieved for values of Young's modulus for a direction of 45°.

Analysing the TSI values summarised in Table 5, we see that with the low TSO values, the divergences between TSI_{MD}, TSI_{MAX} and TSI_{CD}, TSI_{MIN} are minor. In such cases, accepting as the maximum and minimum values of the modulus, the values achieved for MD and CD respectively, does not involve any significant error and is acceptable for practical applications.

After transforming formula (9) and assuming that the constant *c* equals 1, the Young's moduli for MD and CD were calculated on the basis of the results of TSI measurements.

The calculation results, along with the measurements of Young's moduli in the tensile tests, are listed in Table 6. The re-sults show that the Young's modulus values estimated on the base of TSI without constant c involve considerable error.

Conclusions

- The divergence between the values of Young's moduli which were achieved as a result of measurements and calculations confirms that, with the use of low stress appearing at short time intervals, we can use the linear stress theory, treating paper as an orthotropic material.

- In the case of papers for which the value of TSO index is small, we may assume the moduli for longitudinal and cross direction as the maximum and minimum values of Young's moduli. Therefore it can be assumed that the major orthotrophy axes overlap the longitudinal and cross direction in paper.

- When the TSO value is high, the position of the major orthotrophy axes in the paper plane should be determined by the directions of the extreme TSI values.

- Based on measurements of Young's moduli for basic directions and an additional third direction, we can determine the distribution of Young's modulus in the plane of paper accurately enough for practical applications. The highest calculation accuracy can be obtained using for calculating Young's modulus in a direction at angle 45° to the basic directions.

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