Jacek Wiśniewski

Design of Lightweight Composite Disks Reinforced with Continuous Fibres

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Department of Technical Mechanics and Informatics,
Lodz University of Technology
ul. Żeromskiego 116, 90-924 Łódź, Poland
E-mail: jacek.wisniewski@p.lodz.pl

Abstract
The paper presents the results of an investigation in the area of numerical analysis and designing of lightweight composite disks reinforced with continuous fibres. The problem of the optimal layout of reinforcing fibres in the matrix domain in order to obtain the minimal weight of the disk with assumed mechanical properties is considered. The case of the creation of linear and curvilinear fibres is discussed. An adequate model of the composite structure is presented as well as the optimisation task for this type of design problem, and the method of solving this task is formulated in the paper. The design problem is illustrated by a simple numerical example.

Key words: *composite disk, continuous fibres, layout of reinforcement, lightweight structure, optimal design.*

Introduction

Composite materials have a long history of usage. Man was aware, even from the earliest times, of the concept that combining materials could be advantageous. For example, straw was used by the ancient Egyptians to strengthen mud bricks. But it is only in the last half century that composite materials have gained popularity in high-performance products that need to be lightweight and yet strong enough. These materials are mainly meant for the manufacturing of aerospace components, boat hulls, bicycle frames, racing car bodies, sport equipments or products for the protection of human health and life.

The optimal design of composite structures is a very complex process [1, 2]. To fulfil the assumed properties of these structures, a designer can modify some of their structural parameters, such as the properties of matrix and reinforcing fibres, the percentage participation of fibres in the composite material, fibre shape and orientation as well as the stacking sequence, number and thickness. Each of these parameters influences the properties of the composite structure and can be treated as the design variables during the design process.

The weight minimisation of composite structures is the subject of many scientific papers, for example [3 - 8], and this problem constitutes a very important area of research. Many researches have attempted to make better use of material either by minimising the structure thickness or by finding the discrete value of the fibre orientation angle, thus reducing the weight of the composite. However, the intensive development of composite materials technology makes it necessary to find new solutions that can often lead to both cheaper and better results. They can constitute an alternative approach to the existing solutions, or can supplement them.

Object of analysis and basic assumptions

The object of analysis is a thin, two-dimensional and linearly elastic composite disk, shown in ***Figure 1***. The disk has

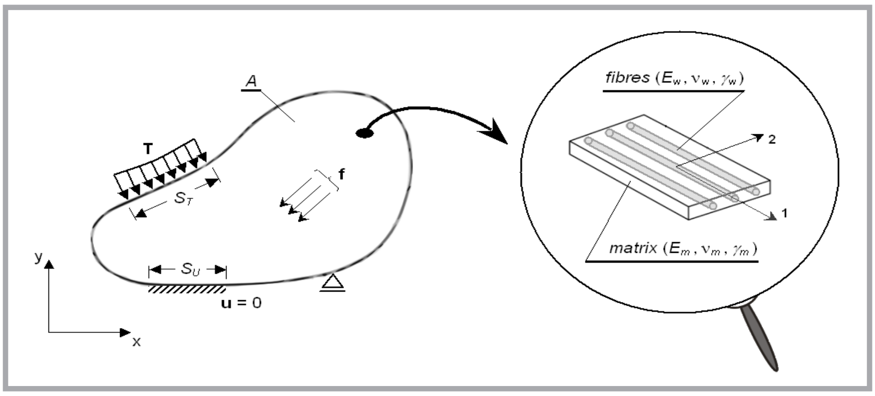

Figure 1. *Composite disk subjected to service load.*

uniform thickness t and is supported on the boundary portion S_U and loaded by a body force **f** within a domain A and by an external traction **T** acting along the boundary portion S_T.

The disk is made of a fibrous composite which consists of a ply of continuous fibres suspended in a matrix. The fibres are the principal reinforcing or load-carrying agent. They are strong and stiff. The function of the light matrix is to support and protect the fibres as well as to provide a means of distributing the load among and transmitting it between the fibres.

The basic assumptions for the components of the composite material are the following:

- The matrix is homogeneous, isotropic and linearly elastic. The density of the matrix is γ_m, and the material properties are characterised by Young's modulus E_m and the Poisson ratio ν_m.
- The fibres are homogeneous, isotropic and linearly elastic and their density is γ_w. They all have the same strength, and Young's modulus and the Poisson ratio of the fibres are denoted by E_w and ν_w, respectively.
- The reinforcing fibres have uniform circular cross-sections and are regularly spaced and perfectly aligned in the matrix.
- The bonds between the matrix and fibres are perfect.

Moreover let us assume that:

- The composite is treated at a macroscopic level as a plane, homogeneous, orthotropic and linearly elastic material.
- The composite is initially stress-free.

Under the load applied, the composite disk undergoes some deformations described by the displacement field **u**, strain field **ε** and stress field **σ**. The behaviour of this structure can be described by an equilibrium equation [9] given in the form:

$$div\,\boldsymbol{\sigma} + \mathbf{f} = 0 \qquad (1)$$

as well as the kinematical relation [9] between strain and displacement fields:

$$\boldsymbol{\varepsilon} = \mathbf{B} \cdot \mathbf{u} \qquad (2)$$

where, **B** is a linear differential operator relating the displacement field with the strain field. A linear stress-strain relation is assumed in the form of generalised Hooke's law [9]:

$$\boldsymbol{\sigma} = \mathbf{D} \cdot \boldsymbol{\varepsilon} \qquad (3)$$

where, **D** denotes the extensional stiffness matrix for the model of the composite material. Besides this, the structure is subjected to the boundary conditions [9] expressed as follows:

$$\begin{cases} \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{T} & \text{on } S_T \\ \mathbf{u} = 0 & \text{on } S_U \end{cases} \qquad (4)$$

where, **n** denotes the normal unit vector on the external boundary S of the disk.

Architecture of fibrous composite

According to the assumptions, a composite is made of two constituents: fibres and a matrix, whose quantities in the material are specified by volume ρ and mass m fractions as follows:

$$\rho_w = \frac{V_w}{V_c}; \quad \rho_m = \frac{V_m}{V_c}$$
$$and \quad \rho_w + \rho_m = 1$$
$$m_w = \frac{M_w}{M_c}; \quad m_m = \frac{M_m}{M_c} \qquad (5)$$
$$and \quad m_w + m_m = 1$$

Notations V and M are the volume and mass, whereas subscripts w, m and c correspond to the fibres, matrix and composite material, respectively. In the problem discussed in this paper, volume fractions will be used because they enter the stiffness coefficients for a lamina. Furthermore mass fractions are usually measured directly during processing or an experimental study of the material fabricated.

The theoretical density of a composite material γ_c can be calculated using the following equation:

$$\gamma_c = \rho_w \left(\gamma_w - \gamma_m\right) + \gamma_m \qquad (6)$$

Where, γ_w and γ_m are the densities of the fibres and the matrix used for fabrication of the composite material. It must be added, that this result is approximate because it ignores possible material porosity.

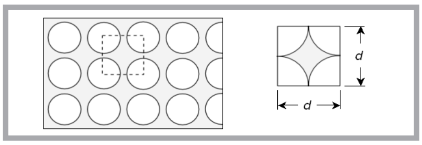

Figure 2. *Ultimate fibre array for square fibre distribution.*

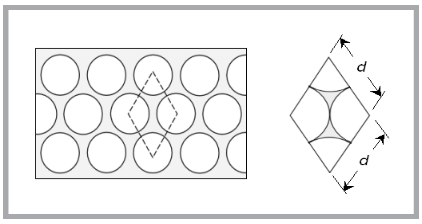

Figure 3. *Ultimate fibre array for hexagonal fibre distribution.*

Since the fibres have uniform circular cross-sections, there exists the ultimate fibre volume fraction $\rho_w{}^u$, which is less than unity and depends on the fibre arrangement. The ultimate fibre arrays for typical idealised regular distributions [10] are presented in ***Figure 2*** and ***Figure 3***, and the corresponding ultimate fibre volume fractions are:

- Square array

$$\rho_w{}^u = \frac{1}{d^2}\left(\frac{\pi d^2}{4}\right) = 0.785 \qquad (7)$$

- Hexagonal array

$$\rho_w{}^u = \frac{2}{d^2\sqrt{3}}\left(\frac{\pi d^2}{4}\right) = 0.907 \qquad (8)$$

The reinforcing fibres can be linearly or curvilinearly spaced in the matrix domain, and their layout at any point of the composite is defined by the fibre orientation angle θ.

In the case of a family of straight fibres (see ***Figure 4***), the fibre orientation angle is constant in the composite domain, and it is directly defined by the angle between the middle line of the so-called di-

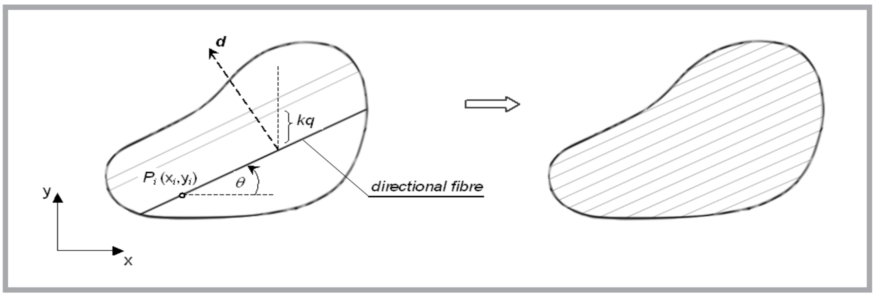

Figure 4. *Composite reinforced with one family of linear fibres.*

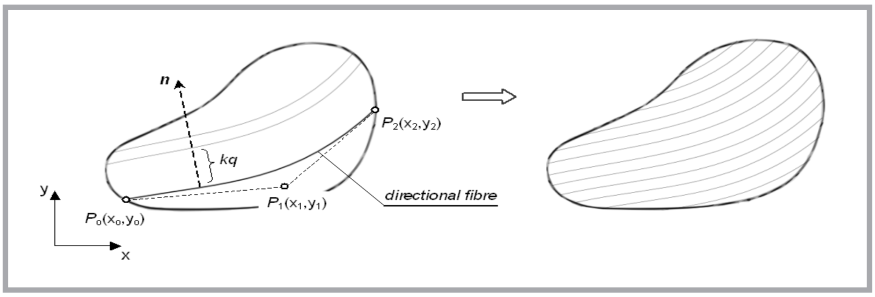

Figure 5. *Composite reinforced with one family of curvilinear fibres.*

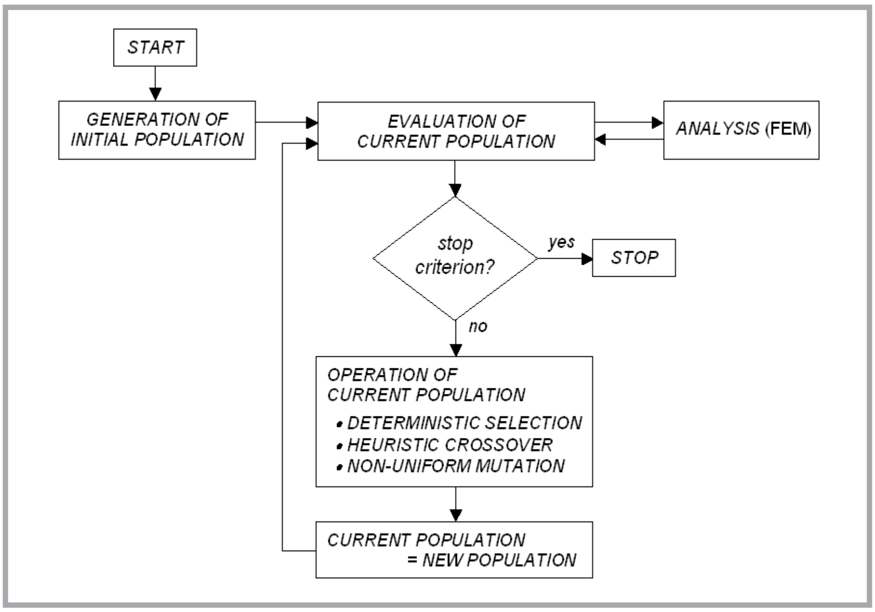

Figure 6. *Flow chart of evolutionary algorithm.*

rectional fibre and the x-axis of the global coordinate system x-y. All other fibres in the family are obtained by translation of the directional fibre in the d-direction, according to the rule:

$$y_i = \mathrm{tg}\ \theta \cdot x_i + kq \qquad (9)$$

where, k is the number of current fibres in the family and q is the distance between two adjacent fibres measured in the y-direction.

Besides this, a family of straight fibres, as well as a family of curvilinear fibres [11, 12], can be created (see ***Figure 5***). In this case, the shape of the directional fibre is defined using Bezier's representation [13] given in the form:

$$\begin{Bmatrix} x(t) \\ y(t) \end{Bmatrix} = \sum_{j=0}^{2} \begin{Bmatrix} x_j \\ y_j \end{Bmatrix} \begin{pmatrix} 2 \\ j \end{pmatrix} t^j (1-t)^{2-j} \qquad (10)$$

where, x_j and y_j are independent coordinates of three nodes of the Bezier polygon. The fibre orientation angle θ varies throughout the composite domain and is now defined by:

$$\mathrm{tg}\,\theta = \left(\frac{y_{,t}}{x_{,t}} \right) \quad \text{and} \quad x_{,t} \neq 0 \qquad (11)$$

where, $y_{,t}$ and $x_{,t}$ denote the derivatives of functions $y(t)$ and $x(t)$ with respect to parameter t, respectively. For the case of curvilinear fibres, the layout of all other fibres in the family is obtained by shifting the directional fibre in the n-direction normal to its middle line:

$$\begin{cases} x_i = x(t) + kq \left(-\dfrac{y_{,t}}{\sqrt{(x_{,t})^2 + (y_{,t})^2}} \right) \\ y_i = y(t) + kq \left(\dfrac{x_{,t}}{\sqrt{(x_{,t})^2 + (y_{,t})^2}} \right) \end{cases} \qquad (12)$$

where, k is the number of current fibres in the family and q is the distance between two adjacent fibres.

Problem formulation

The problem discussed in this paper concerns the design of an optimal fibre arrangement in the matrix domain in order to obtain the minimal weight of the composite disk with the requirements imposed in the range of its mechanical properties. This problem can be formulated as an optimisation task, expressed in a general form:

Weight minimisation of the composite material:

$$\text{min. } F_c = \rho_w \left(\gamma_w - \gamma_m \right) + \gamma_m \qquad (13)$$

subjected to global or local behavioural constraints:

$$\left(\int_A \Gamma(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \mathbf{u}, \mathbf{b})\, dA + \int_{S_T} \Psi(\mathbf{T}, \mathbf{u})\, dS_T \right) - G_0 \leq 0 \qquad (14)$$

Where, Γ and Ψ are continuous functions dependent on the displacement field **u**, strain field **ε** and stress field **σ** induced in the disk subjected to a service load **T** for configuration of the composite material described by design variables **b**. They can be, for instance, a measure of the mean stiffness or compliance of the disk, the strength of the disk etc. Here notation G_0 is the assumed value of these measures.

The geometrical parameters defining the fibre arrangement in the composite, i.e. the fibre orientation angle θ and fibre volume fraction ρ_W will only be treated as design variables in this optimisation task. It must be added that such a defined problem allows to release the designer from the constraints associated with the selection of constituent materials for the composite. The designer can make use of existing conventional materials with properties tailored to suit particular design requirements.

Optimisation strategy

To solve the design problem defined by ***Equations 13 -14***, an optimisation strategy based on the evolutionary algorithm with the penalty function approach is proposed. A flow chart of this strategy is shown in ***Figure 6***, and its detailed description is presented, for instance, in [14].

The evolutionary algorithm starts from a random selection of the initial population of N chromosomes. Each chromosome is a coded vector of design parameters and describes one possible solution to the given problem. This population is processed by three main operators of the evolutionary algorithm: deterministic selection, heuristic crossover and non-uniform mutation. By applying these three operators, a new population of solutions is created and the single cycle of the evolutionary algorithm, which is known as a generation, comes to an end. Each successive generation contains better "partial solutions" than in the previous generations, and converges towards the global optimum. This procedure is continued until the best solution is found according to the assumed stop criterion or the specified number of generations is attained.

As is shown in ***Figure 6***, all chromosomes in the current population are evaluated using the objective functional in the analysis stage of the structural behaviour The set of ***Equations 1 - 4*** is solved with the aid of the finite element method [9] with two-dimensional eight-node quadrilateral elements applied for discretisation of the disk domain.

To analyse the structural behaviour, an adequate model of the fibrous composite must be built. The purpose of the modelling process is to determine the extensional stiffness matrix **D**, appearing in (3), for the composite and to express its components in terms of mechanical and geometrical properties of the matrix and reinforcing fibres.

The extensional stiffness matrix **D** for the assumed model of the composite in the global coordinate system *x-y* can be expressed by [10]:

$$\mathbf{D} = \mathbf{T}^{-1} \cdot \mathbf{C} \cdot \mathbf{T}^{-\mathrm{T}} \tag{15}$$

Here matrix **C** denotes the stiffness matrix for the composite with respect to material axes 1 - 2, coinciding with the fibre direction and the direction perpendicular to the fibre, having the form:

$$\mathbf{C} = \begin{bmatrix} \dfrac{E_1}{1-\nu_{12}\nu_{21}} & \dfrac{E_1\nu_{21}}{1-\nu_{12}\nu_{21}} & 0 \\ \dfrac{E_2\nu_{12}}{1-\nu_{12}\nu_{21}} & \dfrac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \tag{16}$$

where, matrix **T** denotes the transformation matrix from the global coordinate system x-y to the material axes 1 - 2, expressed as ***Equation 17***:

Matrix **T** is considered as the matrix function of fibre orientation angle θ. For the case of straight fibres, its components are explicitly defined by the angle between the fibre line and x-axis of the global coordinate system x-y. When the fibre line is described by (10), sinθ and cosθ follow from (11), expressed in the form:

$$\sin\theta = \frac{tg\,\theta}{\sqrt{1+tg^2\theta}} = \frac{y_{,t}}{\sqrt{(x_{,t})^2+(y_{,t})^2}}$$
$$\cos\theta = \frac{1}{\sqrt{1+tg^2\theta}} = \frac{x_{,t}}{\sqrt{(x_{,t})^2+(y_{,t})^2}} \tag{18}$$

The components of the matrix **C** depend on so-called engineering constants for orthotropic lamina, where E_1 and E_2 are the apparent Young's modules in the fibre direction and the direction perpendicular to the fibre, respectively, while ν_{12} and ν_{21} are the major and minor Poisson's ratios, and G_{12} denotes the in-plane shear modulus.

Approximation of the mechanical properties for different types of fibrous composites is widely discussed in the literature [10, 15, 16]. For the composite considered in this paper, Halpin and Tsai's model, presented in [10], is proposed. Using this model, the engineering constants have the ***Equation 19*** form.

Here parameter ξ is a measure of the fibre reinforcement of the composite dependent on the cross-section of fibres and packing geometry [10].

$$\mathbf{T} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta-\sin^2\theta \end{bmatrix} \tag{17}$$

$$\begin{aligned} &E_1 = E_w\rho_w + E_m(1-\rho_w) \\ &E_2 = E_m\left(\frac{1+\xi\chi\rho_w}{1-\chi\rho_w}\right) \quad \text{where:} \quad \chi = \frac{\left(E_w/E_m\right)-1}{\left(E_w/E_m\right)+\xi} \\ &\nu_{12} = \nu_w\rho_w + \nu_m(1-\rho_w) \\ &\nu_{21} = \frac{E_2}{E_1}\nu_{12} \\ &G_{12} = G_m\left(\frac{1+\xi\chi\rho_w}{1-\chi\rho_w}\right) \quad \text{where:} \quad \chi = \frac{\left(G_w/G_m\right)-1}{\left(G_w/G_m\right)+\xi} \end{aligned} \tag{19}$$

Equations 17 and 19.

Table 1. *Properties of aluminium and components of the composite.*

Material	E, GPa	ν	γ, 10^3 kg/m^3
Aluminium	72.0	0.33	2.80
Epoxy matrix	3.5	0.38	1.15
Graphite fibres	230.0	0.25	1.74

Numerical Example

To illustrate the problem of the optimal design and analysis of lightweight composite disks subjected to a service load, a simple example (see ***Figure 7***) is pre-

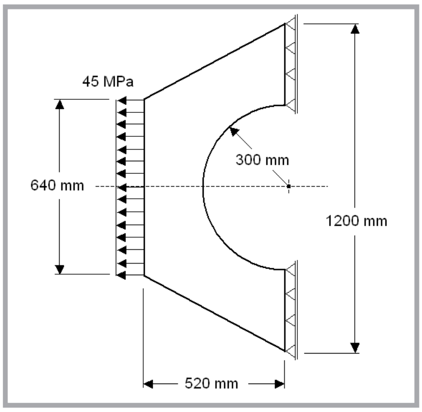

Figure 7. *Culvert subjected to load and boundary conditions.*

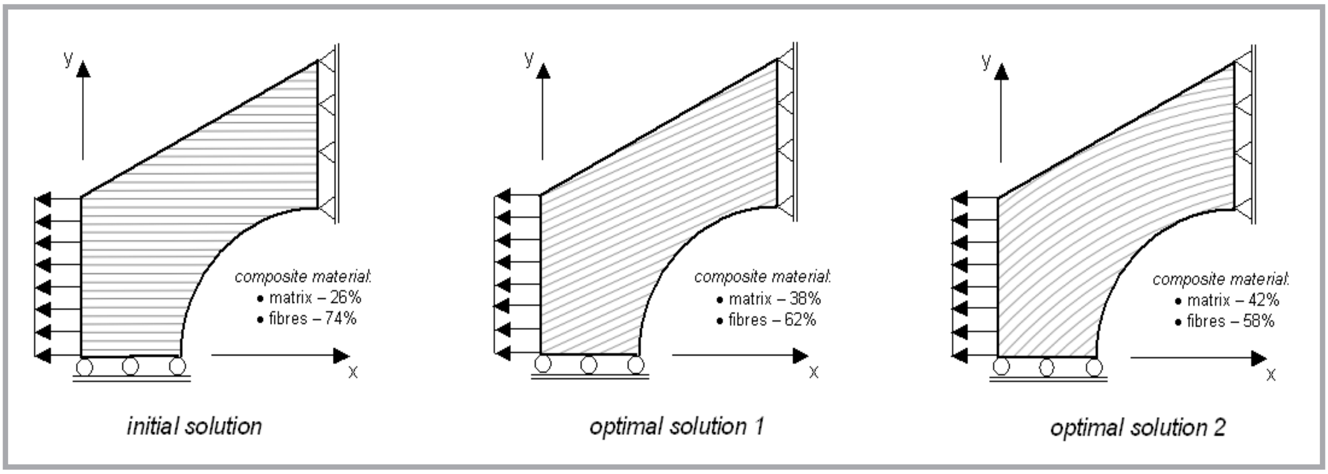

Figure 8. *Layout of reinforcing fibres in the culvert.*

sented in this Section. A thin, two-dimensional culvert with such a load can exist anywhere in a technical space.

Let us assume that the culvert is made of aluminium (see ***Table 1***), which is one of the more lightweight construction materials. On the basis of numerical analysis of this structure, it has been established that its mean stiffness corresponding to the work performed by the external traction on the actual displacements is 4.40 in J, i.e.:

$$STIFF = \int_{S_T} \mathbf{u}^{\mathrm{T}} \cdot \mathbf{T}\, dS_T = 4.40 \text{ J} \quad (20)$$

Next aluminium material is replaced with a composite material made of an epoxy matrix reinforced with graphite fibres (see ***Table 1***) in order to minimise the weight of the culvert with imposed requirements in the range of its mean stiffness. A family of straight fibres parallel to the x-axis for idealised square distribution in the cross section of a ply is considered in the initial solution (see ***Figure 8***). The result of the analysis of this structure is given in ***Table 2***. One can easy observe that the weight of a culvert made of the composite material is considerably reduced and its mean stiffness is the same. However, this initial solution with a classical layout of reinforcing fibres in the matrix domain can be not optimal.

To find an optimal solution, the composite structure must be subjected to the optimisation process, as follows:

- the weight minimisation of the composite

$$\rho_w \left(\gamma_w - \gamma_m\right) + \gamma_m \rightarrow \min. \quad (21)$$

- subjected to global behavioural constraints:

$$\int_{S_T} \mathbf{u}^{\mathrm{T}} \cdot \mathbf{T}\, dS_T - 4.40 \le 0 \quad (22)$$

The problem was discussed for two classes of layout of the unidirectional fibres in the matrix domain, i.e. a family of linear and curvilinear reinforcement. Thus the fibre shape parameters defining thist particular representation and fibre volume fraction were treated as design variables. In view of the geometry of the structure, the variability intervals for these parameters were as follows:

- family of linear fibres

$$0 < \rho_W \le 0.785 \text{ and } 0 \le \theta \le 180° \quad (23)$$

- family of curvilinear fibres

$$0 < \rho_W \le 0.785 \text{ and} \quad (24)$$

$$\begin{cases} x0 = 0 & 0 \le y0 \le 0.6 \\ 0 < x1 < 0.52 & 0 \le y1 \le 0.6 \\ y2 = 0.52 & 0 \le y2 \le 0.6 \end{cases}$$

The results obtained after the optimisation process for these cases are given in ***Table 2***, and the optimal layouts of fibres in the half-symmetry model of the culvert are depicted in ***Figure 8***, respectively. The optimal layout in the case of the family of straight fibres decreases the weight of the disk by 5%, while in the case of the family of curvilinear fibres, this weight decreases by 7%, when compared to the initial solution.

Table 2. *Results of optimisation process and reference solutions*

	Fibre shape parameters	Fibre volume fraction	Density, 10^3 kg/m^3	Stiffness, J
Aluminium	–	–	2.80	4.40
Initial solution	θ = 0°	ρ_W = 0.74	1.59	
Optimal solution 1	θ = 24.21°	ρ_W = 0.62	1.52	
Optimal solution 2	P_0 (0; 0.16) P_1 (0.22; 0.46) P_2 (0.52; 0.31)	ρ_W = 0.58	1.49	

Conclusions

The optimal design of composite structures is a very complex process. To fulfil the minimal weight of composite disks subjected to a service load, a designer can modify some mechanical and geometrical properties of components of the composite material. However, as shown in this paper, optimal solutions can be obtained when reinforcing fibres are optimally distributed and oriented in the matrix domain. Such an optimisation problem solution allows to release a designer from the constraints associated with the selection of constituent materials of a composite. The designer can make use of existing conventional materials with properties tailored to suit particular design requirements.

The paper is purely theoretical and presents the results of an initial investigation in the area of the weight minimisation of disks made of composite lamina. The solution proposed can be treated as a starting point for further analysis of various types of composite structures, such as composite disks reinforced with discontinuous fibres or in laminates with different layouts.

After practical verification of the results of the numerical analysis, the solution proposed can be used in a computer-oriented design procedure for real composite structures subjected to a particular load. Such a procedure can allow to avoid expensive experimental testing, which can be reduced to the final phase of structural design. However, due of a lack of opportunities for verification, this stage of validation is beyond the problem presented and can be discussed within consecutive work.

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